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COMPACT SETS DEFINABLE BY FREE 3-MANIFOLDS

BY

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ABSTRACT. Shape conditions are given that force a compactum (i.e., a compact metric space) embedded in the interior of a nonclosed, piecewise-linear 3-manifold to have arbitrarily close, compact, polyhedral neighborhoods each component of which is a 3-manifold with free fundamental group (i.e., to be definable by free 3-manifolds). For compact, connected ANR's these conditions reduce to the criterion of having a free fundamental group. Additional conditions are given that insure definability by handlebodies or cubes-with-handles. An embedding of Menger's universal 1-dimensional curve in Euclidean 3-space is shown to have the property that all tame surfaces, separating in 3-space a fixed pair of points, cannot be adjusted (by a small space homeomorphism) to intersect the embedded curve in a 0-dimensional set.

1. Introduction. Let X be a compact metric space and let h be a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. We say $h(X)$ is *definable by free 3-manifolds* if $h(X) = \bigcap_{i=1}^{\infty} H_i$ where H_i is a compact polyhedron in Int M^3 (the interior of M^3), each component of H_i is a 3-manifold with free fundamental group, and $H_{i+1} \subset \text{Int } H_i$. Keep this notation fixed throughout the introduction.

In general the embedding h may determine if $h(X)$ is definable by free 3-manifolds. (See the examples in [23], [4], [5].) This paper gives shape properties of the components of X that imply $h(X)$ is definable by free 3-manifolds. (§4, which deals with properties of special embeddings of 1-dimensional compacta, is an exception.) Our properties (uv(r, s; G) and UVF defined in §2) fall under the category of "UV" or shape properties. We have chosen to define them in shape terms (i.e., ANR-sequences) rather than by neighborhood pairs (as in [19]). For embeddings in 3-manifolds these alternatives are equivalent.

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Let us name some special 3-manifolds with free fundamental group. A compact, piecewise-linear 3-manifold H^3 is a *handlebody* if H^3 is a regular neighborhood of a finite, polyhedral, connected graph in Int H^3. An orientable handlebody is a *cube-with-handles*. We say $h(X)$ is *definable by cubes-with-handles* if $h(X)$ is definable by free 3-manifolds and in addition each component of H_i (in the definition of definable by free 3-manifolds) is a cube-with-handles. Similarly we can define *definable by handlebodies*, *definable by 3-cells*, etc.

McMillan has studied the property of being definable by cubes-with-handles rather extensively [16], [17], [18], [19], [20], [21], [22]. If M^3 is orientable and X is a 1-dimensional ANR [1], [16], a polyhedron collapsible to a 1-dimensional subpolyhedron [16], a UV^∞ set [19], or a Sierpiński curve [22], then $h(X)$ is definable by cubes-with-handles. Theorem 8.1 of Craggs [6] implies that if X is a polyhedron each component of which has free fundamental group, then $h(X)$ is definable by free 3-manifolds. (Craggs actually obtains a much stronger result.) McMillan [17] had previously shown that if X is a 2-sphere then $h(X)$ is definable by free 3-manifolds. In [18], [19] McMillan provides acyclicity conditions that insure $h(X)$ is definable by homotopy cubes-with-handles.

Theorem 15, and Corollaries 14, 16, 17, 19, 20 contain the results alluded to in the abstract. Our main result, Theorem 15, can be regarded as a generalization of [18, Theorem 3]. If we just consider the question of $h(X)$ being definable by free 3-manifolds, McMillan's results on planar compacta [22] are the only results mentioned that lie outside the scope of Theorem 15.

Let n be a positive integer. If the components of X are 2-spheres, diadic solenoids, toroidal continua, or are definable by cubes with no more than n handles, then Theorem 15 implies that $h(X)$ is definable by free 3-manifolds. Another special case of the following question is answered in [22]. If each component of a compact set Y embedded in the interior of a 3-manifold is definable by free 3-manifolds, is Y definable by free 3-manifolds?

We will assume the reader is acquainted with regular neighborhood theory [28] and geometric group theory [12], [15]. A good reference for algebraic topology is [25].

We will use Z_p, p a prime or zero, to denote the integers modulo p (Z_0 is the integers). We only use Z_p as coefficient group for homology groups. In a given context p will remain fixed. (We also use Z_i, Z_0, Z as names for polyhedra. There should be no confusion.)

We adopt the convention that unidentified arrows between homology or homotopy groups will denote inclusion induced homomorphisms. If $A \to B$ is a homomorphism we denote the kernel and image of $A \to B$ by $\ker[A \to B]$, image$[A \to B]$, respectively. We use ρG to denote the rank of a group G (see §2 for the definition). Manifolds are always connected, separable, metric spaces. We use ∂M to denote the boundary of a manifold M.

2. Shape properties. In this section we will show properties uv(r, s; G) and UVF, defined below, are shape properties for compact metric spaces. We will use the ANR-sequence approach to shape developed in [13], [14]. See p. 61 of [13] for a complete set of the definitions involved and a comparison with Borsuk's definition of shape [3].

Let X be a compact metric space. Then $\underline{X} = \{X_n, p_{n,n+1}\}$ is *an* ANR-*sequence associated with* X if $\underline{X}$ is an inverse sequence of compact ANR's for metric spaces and X is homeomorphic to the inverse limit of $\underline{X}$. We will suppress the homeomorphism between X and the inverse limit of $\underline{X}$. A *map of sequences* $\underline{f}$: $\underline{X} \to \underline{Y}$ where $\underline{Y} = \{Y_n, g_{n,n+1}\}$ consists of an increasing function f: $N \to N$ (N is the natural numbers) and a collection of maps f_n: $X_{f(n)} \to Y_n$ such that $f_n p_{f(n),f(n')}$ is homotopic to $g_{n,n'} f_{n'}$ for $n \leq n'$. Two maps $\underline{f}$, $\underline{g}$: $\underline{X} \to \underline{Y}$ are *homotopic*, $\underline{f} \simeq \underline{g}$, provided for each n in N there is an n' in N, $n' \geq f(n)$, $g(n)$, such that $f_n p_{f(n),n'} \simeq g_n p_{g(n),n'}$. The composite $\underline{gf}$: $\underline{X} \to \underline{Z}$ of $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{Z}$ is a map of sequences $\underline{h}$: $\underline{X} \to \underline{Z}$ given by $h = fg$: $N \to N$ and $h_n = g_n f_{g(n)}$: $X_{fg(n)} \to Z_n$. The identity map of sequences $\underline{1}_X$: $\underline{X} \to \underline{X}$ is given by 1_N: $N \to N$ and 1_{X_n}: $X_n \to X_n$. Two metric compacta X and Y have the same *shape*, $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, provided for some ANR-sequences $\underline{X}$ and $\underline{Y}$ associated with X and Y, respectively, there exist maps of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ such that $\underline{gf} \simeq \underline{1}_X$ and $\underline{fg} \simeq \underline{1}_Y$. We say $\mathrm{Sh}(X)$ *fundamentally dominates* $\mathrm{Sh}(Y)$, written $\mathrm{Sh}(X) \geq \mathrm{Sh}(Y)$, if in the above definition of $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ we do not require $\underline{gf} \simeq \underline{1}_X$. Borsuk defines $\mathrm{Sh}(X) \geq \mathrm{Sh}(Y)$ on p. 25 of [3] and the proofs in [13] show the definitions are equivalent for compact metric spaces. In [14] Mardešić and Segal show the notions of shape and fundamental domination do not depend on our choices of $\underline{X}$ and $\underline{Y}$ associated with X and Y.

We will need the following items to define properties uv(r, s; G) and UVF.

If A is a finitely generated group let the *rank of* A, written ρA, be the least number of generators in a presentation of A. We will use the following properties of rank. (J. E. Leech provided the proof of (4).) (1) If A is a finitely generated group and f: $A \to B$ is an epimorphism then $\rho B \leq \rho A$. (2) If A and B are finitely generated abelian groups and f: $A \to B$ is an epimorphism then $\rho A \leq \rho B + \rho \ker f$. (3) If A is a finite dimensional vector space over Z_p, p a prime, then $\rho A = \dim A$. (4) If A is a finitely generated abelian group and B is a subgroup of A then $\rho B \leq \rho A$. (Note that each epimorphism from the free group on k elements onto A factors through the free abelian group G_k on k elements. If we consider the preimage of B in G_k, we see that $\rho B \leq k$.)

If d: $A \to B$ is a group homomorphism we say d *factors through a free group* if there exist a free group F and homomorphisms d_1: $A \to F$, d_2: $F \to B$ such that $d_2 d_1 = d$. If X is a topological space, frequently we will use a prime, X', to denote a component of X. If f: $X \to Y$ is a map then f': $X' \to Y'$ will denote the map $f|_{X'}$: $X' \to Y'$ where Y' is the component of Y that contains $f(X')$.

Let $\underline{X} = \{X_n, p_{n,n+1}\}$ be an ANR-sequence associated with a compact metric space X. Let G be a finitely generated Abelian group. Suppose r and s are integers, $r \geq 1$, and $s \geq 0$. We say X has *property* uv$(r, s; G)$ if given k there exists l such that if X_l' is a component of X_l and $(p_{k,l}')_*$: $H_r(X_l'; G) \to H_r(X_k'; G)$ is the homomorphism induced by $p_{k,l}'$, then

$$\rho(p_{k,l}')_*(H_r(X_l'; G)) \leq s.$$

X has *property* UVF if given k there exists l such that if X_l' is a component of X_l then the homomorphism $(p_{k,l}')_\#$: $\pi_1(X_l', x) \to \pi_1(X_k', p_{k,l}'(x))$, induced by $p_{k,l}'$, factors through a free group. Note that any map f: $X_l' \to X_k'$ that is homotopic (not necessarily keeping x fixed) to $p_{k,l}'$ also induces a homomorphism that factors through a free group. Theorem 1 shows properties uv$(r, s; G)$ and UVF do not depend on our choice of $\underline{X}$.

Remark. Suppose h is a topological embedding of a compact metric space X into the interior of a nonclosed, piecewise-linear 3-manifold M^3 and $H_1, \cdots, H_n, \cdots$ are compact polyhedra in Int M^3 such that $h(X) = \bigcap_{n=1}^{\infty} H_n$, each component of H_n is a 3-manifold, and $H_{n+1} \subset \text{Int } H_n$. Let $h_{n,n+1}$: $H_{n+1} \to H_n$ denote the inclusion map. Our primary interest is in the ANR-sequence $\underline{H} = \{H_n, h_{n,n+1}\}$ associated with X.

Note our property uv$(n, 0; Z_p)$ is equivalent to McMillan's property $n -$ uv(Z_p) [19] for embeddings in 3-manifolds. By using a p-adic (p a prime) solenoid type construction with an increasing number of handles we can find a continuum that has property uv$(1, 0; Z_p)$ but not property uv$(1, k; Z_q)$ for each k and prime $q \neq p$.

Theorem 1. *Properties* uv$(r, s; G)$ *and* UVF *are shape properties and are preserved by fundamental domination.*

Proof. It suffices to show our properties are preserved by fundamental domination. Suppose Sh$(X) \geq$ Sh(Y) where X and Y are compact metric spaces. Using the notation introduced in defining Sh$(X) \geq$ Sh(Y), we have maps of sequences $\underline{f}$: $\underline{X} \to \underline{Y}$ and $\underline{g}$: $\underline{Y} \to \underline{X}$ such that $\underline{fg} \simeq \underline{1}_Y$. It follows from the definitions that given k, for all $l \geq f(k)$ there exists an $n \geq g(l)$ such that

$$q_{k,n} \simeq f_k p_{f(k),l} g_l q_{g(l),n}$$

(i.e., $q_{k,n}$ homotopy factors through $p_{f(k),l}$).

Suppose X has property uv$(r, s; G)$. Then given k, for some l and all components X_l' of X_l, $\rho(p_{f(k),l}')_*(H_r(X_l'; G)) \leq s$. Hence there exists an $n \geq g(l)$ such that for each component Y_n' of Y_n, $\rho(q_{k,n}')_*(H_r(Y_n'; G)) \leq s$. So Y has property uv(r, s; G).

Suppose X has property UVF. Then given k, for some l and all components X_l' of X_l, $(p_{f(k),l}')_\#$ factors through a free group. Hence there exists an $n \geq g(l)$ such that for each component Y_n' of Y_n, $(q_{k,n}')_\#$ factors through a free group. Therefore Y has property UVF.

An easy compactness argument yields

Lemma 2. *Suppose X is a compact metric space and* P *is one of the properties* uv(r, s; G) *or* UVF. *Then X has property* P *if and only if each component of X has property* P.

Theorem 3. *Suppose X is a compact metric space and each component of X has property* uv(2, s; Z_p) *where $p = 0$ or a prime. Let h be a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. If M^3 is nonorientable assume $p = 2$. Then $h(X) = \bigcap_{i=1}^\infty H_i$ where H_i is a compact polyhedron in* Int M^3, *each component of H_i is a 3-manifold with no more than $s + 1$ boundary components, and $H_{i+1} \subset$* Int H_i.

Proof. Let U be a neighborhood of $h(X)$ in Int M^3. It is sufficient to find a compact polyhedron $H \subset U$ such that each component of H is a 3-manifold with no more than $s + 1$ boundary components and $h(X) \subset \text{Int } H$. Since X has property uv(2, s; Z_p) we can find compact polyhedra P and R contained in U such that each component of P or R is a 3-manifold, $P \subset \text{Int } R$, $h(X) \subset \text{Int } P$, and for each component P' of P and R' of R such that $P' \subset R'$ we have

$$\rho \text{ image} [H_2(P'; Z_p) \rightarrow H_2(R'; Z_p)] \leq s.$$

Suppose P' is a component of P and P' is contained in R', a component of R. Let H' be obtained from P' by adding to P' all components of the closure of $R' - P'$ that do not intersect $\partial R'$, the boundary of R'. Note $H' \supset P'$, $\partial H' \subset \partial P'$, each component of $R' - H'$ meets $\partial R'$, and if P'' is another component of P then H' contains P'' or $H' \cap P'' = \emptyset$.

H' has at most $s + 1$ boundary components. To establish this claim let S_1, $\cdots$, S_r be the boundary components of H'. Let $J_1, \cdots, J_{r-1}$ be polyhedral arcs in R' such that J_i pierces each of S_i and S_r in exactly one point, $J_i \cap S_j = \emptyset$ for $j \neq i$ or r, and $\partial J_i \subset \partial R'$. There exist homomorphisms d_i: $H_2(R'; Z_p) \rightarrow Z_p$ given by computing the geometric intersection number mod p of a representative of an element of $H_2(R'; Z_p)$ with J_i. Hence $S_1, \cdots, S_{r-1}$ are carriers of independent elements of $H_2(R'; Z_p)$. Since $\partial H' \subset \partial P'$, $r - 1 \leq s$. Hence $r \leq s + 1$ as claimed.

If H' and H'' are obtained from components P' and P'' of P, respectively, as in the preceding two paragraphs then $H' \cap H'' = H'$, H'', or $\emptyset$. Suppose not. Since

$P' \cap P'' = \emptyset$ there exist components Q' of the closure of $R' - P'$ and Q'' of the closure $R'' - P''$ such that $Q' \cap Q'' \neq \emptyset$, $\partial Q' \subset P'$, $\partial Q'' \subset P''$, $P' \subset Q''$, and $P'' \subset Q'$. Hence $Q' \cup Q''$ is a closed 3-manifold contained in M^3, an impossibility.

Now a maximal disjoint collection of the 3-manifolds H' will be the components of the required H.

3. Simple moves in 3-manifolds. In this section we will develop the algebraic-geometric background we need. This section relies heavily on results of D. R. McMillan, Jr.

We will assume the reader is familiar with McMillan's notation and results on simple moves in an orientable, nonclosed, piecewise-linear 3-manifold M^3 (see §2 of [18] and [20]). If we remove the requirement that M^3 be orientable, only the definition of $c(Z^3)$, on p. 130 of [18] and p. 162 of [20], needs to be altered. Redefine $c(Z^3) = \Sigma\,(3 - \chi(S))^2$ where the sum extends over all closed surfaces S in ∂Z^3 and $\chi(S)$ is the Euler characteristic of S. Now McMillan's results in §2 of [18] and §2 of [20] hold for possibly nonorientable, nonclosed, piecewise-linear 3-manifolds M^3.

We will basically retain McMillan's notation in this section. Note 3-manifolds are connected and M^3 will always be a nonclosed, piecewise-linear 3-manifold. We deviate from McMillan's notation by using superscripts on 3-manifolds (i.e., M^3) and dropping them on compact polyhedra each component of which is a 3-manifold (i.e., W_i instead of W_i^3). Recall that unidentified arrows denote inclusion induced homomorphisms.

Lemma 4. *Let $p = 0$ or a prime. Suppose W_1, W_2, W_3 are compact polyhedra such that each component of W_i is a 3-manifold, $W_3 \subset \operatorname{Int} W_2$, and $W_2 \subset \operatorname{Int} W_1$. Suppose that W is obtained from W_2 by an annexation of type 2 or a simple reduction in W_1 and that $W_3 \subset \operatorname{Int} W$. Let W' be a component of W and suppose $W_3' \subset W_2' \subset W_1'$ are components of W_3, W_2, W_1, respectively, such that $W_3' \subset W'$. Then*

$$\rho \text{ image } [H_1(W_3'; Z_p) \to H_1(W'; Z_p)] \leq \rho \text{ image } [H_1(W_3'; Z_p) \to H_1(W_2'; Z_p)]$$

and $\text{image}[H_1(W'; Z_p) \to H_1(W_1'; Z_p)]$ *is a subgroup of* $\text{image}[H_1(W_2'; Z_p) \to H_1(W_1'; Z_p)]$.

Proof. If $W' = W_2'$ we are done. So suppose $W' \neq W_2'$. If W is obtained from W_2 by an annexation of type 2 then using the Mayer-Vietoris sequence of W_2' and the closure of $W' - W_2'$ with coefficients Z_p [25, p. 218] we see that $H_1(W_2'; Z_p) \to H_1(W'; Z_p)$ is an epimorphism. If W is obtained from W_2 by simple reduction then using the Mayer-Vietoris sequence of W' and the closure of $W_2' - W'$ with coefficients Z_p we see that $H_1(W'; Z_p) \to H_1(W_2'; Z_p)$ is a monomorphism. These two facts applied to the four appropriate diagrams yield the desired conclusions.

Lemma 5 for orientable 3-manifolds is given in [21, Lemma 10.2]. We use essentially the same proof. It is included here for completeness.

Lemma 5. *Let* M^3 *be a compact* 3-*manifold with nonempty boundary. Let* p *be a prime and if* M^3 *is nonorientable assume* $p = 2$. *Then*

$$2\rho \text{ image } [H_1(\partial M^3; Z_p) \to H_1(M^3; Z_p)] = \rho H_1(\partial M^3; Z_p).$$

Proof. Let $I = \text{image}[H_1(\partial M^3; Z_p) \to H_1(M^3; Z_p)]$. Note M^3 is Z_p-orientable. The exact homology sequence (Z_p coefficients) of the pair $(M^3, \partial M^3)$ may be separated into two exact sequences thus:

$$\begin{array}{ccccccccccccc} 0 \to & H_3(M, \partial M) & \to & H_2(\partial M) & \to & H_2(M) & \to & H_2(M, \partial M) & \to & H_1(\partial M) & \to I & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\ 0 \leftarrow & H_0(M) & \longleftarrow & H_0(\partial M) & \leftarrow & H_1(M, \partial M) & \longleftarrow & H_1(M) & \longleftarrow & I & \longleftarrow 0 & . \end{array}$$

The vertical arrows represent isomorphisms given by Lefschetz duality [25, p. 298] and the universal coefficient theorem [25, p. 244]. By exactness the alternating sum of the ranks of the groups appearing in each row is zero. Hence our conclusion follows.

Corollary 6 for orientable 3-manifolds appears in [21].

Corollary 6. *Suppose* T *is a boundary component of a compact* 3-*manifold* M^3. *Let* p *be a prime and if* M^3 *is nonorientable assume* $p = 2$. *Then*

$$2\, \rho \text{ image } [H_1(T; Z_p) \to H_1(M^3; Z_p)] \geq \rho\, H_1(T; Z_p).$$

Proof. In this proof homology groups will be with Z_p coefficients. Let D^3 be the 3-manifold obtained by taking disjoint copies M_1^3, M_2^3, of M^3 and identifying corresponding boundary components by the identity map except for T_1 and T_2, the copies of T. Suppose $2\rho \text{ image}[H_1(T) \to H_1(M^3)] < \rho H_1(T)$. Then

$$\begin{aligned} 2\rho \text{ image } [H_1(T_1 \cup T_2) \to H_1(D^3)] &\leq 4\rho \text{ image } [H_1(T_1) \to H_1(D^3)] \\ &\leq 4\rho \text{ image } [H_1(T) \to H_1(M^3)] \\ &< 2\rho H_1(T) = \rho H_1(T_1 \cup T_2), \end{aligned}$$

which contradicts Lemma 5.

See the proof of [18, Theorem 2] for more details of the proof of Theorem 7.

Theorem 7. *Suppose* X *is a compact subset of the interior of a nonclosed, piecewise-linear* 3-*manifold* M^3. *Suppose each component of* X *has property* uv$(1, m; Z_p)$ *where* p *is a prime or zero and* $p = 2$ *if* M^3 *is nonorientable. Then there exists a compact polyhedron* Z *in* Int M^3 *such that each component of* Z *is a* 3-*manifold*, $X \subset \text{Int } Z$, *and there exists a* Z_0 *obtained from* Z *by simple moves in* M^3 *such that each component* T *of* ∂Z_0 *satisfies* $\rho H_1(T; Z_p) = \rho H_1(T; Z_2) \leq 2m$.

Proof. We may suppose M^3 is compact. Assume p is a prime. According to the Finiteness Theorem of Haken [7], [8], there exists a positive integer H such that any collection of H or more disjoint, incompressible, polyhedral surfaces in Int M^3 contains a pair of surfaces that are topologically parallel. Using an ANR-sequence for X in M^3 of the type described in the remark in §2, we can find compact polyhedra $Z_1, \cdots, Z_{H+1}$ such that

(1) each component of Z_i is a 3-manifold, $i = 1, \cdots, H+1$,

(2) $Z_1 \subset \operatorname{Int} M^3$ and $Z_{i+1} \subset \operatorname{Int} Z_i$, $i = 1, \cdots, H$,

(3) if Z'_{i+1} is a component of Z_{i+1} and Z'_i is the component of Z_i that contains Z'_{i+1} then ρ image$[H_1(Z'_{i+1}; Z_p) \to H_1(Z'_i; Z_p)] \leq m$, $i = 1, \cdots, H$, and

(4) $X \subset \operatorname{Int} Z_{H+1}$.

Call an $(H+1)$-tuple $\{Z_1, \cdots, Z_{H+1}\}$ *admissible* if it satisfies (1), (2), and (3). Lemma 4 shows that if $Z_i^{\#}$ is obtained from Z_i by a simple annexation of type 2 or a simple reduction in Z_{i-1} and $Z_{i+1} \subset \operatorname{Int} Z_i^{\#}$, then $\{Z_1, \cdots, Z_i^{\#}, \cdots, Z_{H+1}\}$ is admissible. (The exceptional cases $i = 1$ or $H+1$ are left to the reader.) Hence there is an admissible $(H+1)$-tuple $\{Z_1^*, \cdots, Z_{H+1}^*\}$, obtained from $\{Z_1, \cdots, Z_{H+1}\}$ by simple annexations of type 2 or simple reductions, that admits no further such moves. Any boundary component of Z_i^* that is not a 2-sphere must be incompressible. (See the proof of [18, Theorem 2].)

If some Z_i^* has only boundary components of rank $\leq 2m$, that Z_i^* is our required Z_0 for $Z = Z_i$. So suppose each Z_i^* has a boundary component T_i such that $\rho H_1(T_i; Z_p) > 2m$. Now $T_1, \cdots, T_{H+1}$ are disjoint, incompressible, polyhedral surfaces in Int M^3 so some T_i, T_j, $i < j$, bound a parallelity component A in M^3 (i.e., A is piecewise-linearly homeomorphic to $T_i \times [0, 1]$ and $\partial A = T_i \cup T_j$). By the Lemma in [7, Appendix] we can assume $j = i + 1$ (we may need to replace T_k by another boundary component of Z_k^*) and that Int $A \cap (\bigcup_{i=1}^{H+1} \partial Z_i^*)$ consists entirely of 2-spheres. Since A is irreducible, A is contained in Z_i^* except for the interiors of finitely many disjoint 3-cells contained in Int A. Let $Z_k^{*\prime}$ be the component of Z_k^* that contains T_k, $k = i$ or $i+1$. Hence

$$\rho \text{ image}[H_1(Z^{*\prime}_{i+1}; Z_p) \to H_1(Z^{*\prime}_i; Z_p)] \geq \rho \text{ image}[H_1(T_{i+1}; Z_p) \to H_1(Z^{*\prime}_i; Z_p)]$$

$$= \rho \text{ image}[H_1(T_i; Z_p) \to H_1(Z^{*\prime}_i; Z_p)] > m.$$

The last inequality follows from Corollary 6. But now the admissibility of $\{Z_1^*, \cdots, Z_{H+1}^*\}$ is violated. The proof is complete for p a prime once we show that if T is an orientable surface, $\rho H_1(T; Z_p) = \rho H_1(T; Z_2)$. But $H_1(T; Z)$ is a free abelian group of rank twice the genus of T [15, p. 132]. So by the universal coefficient theorem, $\rho H_1(T; Z) = \rho H_1(T; Z_p) = \rho H_1(T; Z_2)$.

Suppose $p = 0$. But uv$(1, m; Z)$ implies uv$(1, m; Z_2)$. Hence the above proof yields the desired conclusion. The proof of Theorem 7 is complete.

Theorem 8. *Let M^3 be a nonclosed piecewise-linear 3-manifold. Let W_i, $i = 1, \cdots, k$, be compact polyhedra in* Int M^3 *such that each component of W_i is a 3-manifold and $W_{i+1} \subset$* Int W_i. *Let Z be a compact polyhedron in* Int W_k *such that each component Q^3 of Z is a 3-manifold that has free image in some W_i (i.e., there exists a component R^3 of W_i such that $Q^3 \subset R^3$ and* image$[\pi_1(Q^3) \to \pi_1(R^3)]$ *is a free group). Then by applying extended simple moves to Z in M^3 we can obtain a Z_0 such that each component of Z_0 is simply connected.*

Proof. Let l be the smallest integer such that each component of Z has free image in some W_s, $s \leq l$. We will induct on l.

Suppose $l = 1$. Apply Theorem 1 of [20] to each component of W_1 with P being the property of being a trivial group.

Suppose $l > 1$. There exists a Z_1 obtained from Z by extended simple moves in W_l and such that Z_1 admits no further extended simple moves in W_l. Let Q_1^3 be a nonsimply connected component of Z_1. Let Q^3 be Q_1^3's ancestor in Z. By [20, Lemma B$^+$] since Q^3 has free image in W_s, $s \leq l$, Q_1^3 has free image in W_s But if Q^3 has free image in W_l then Theorem 1 of [20] implies that Q_1^3 is simply connected. Hence each component of Z_1 has free image in some W_s where $s \leq l - 1$. By inductive hypothesis there is a Z_0 obtained from Z_1 by extended simple moves in M^3 such that each component of Z_0 is simply connected. Z_0 is our required polyhedron.

Caution. Theorem 8 asserts only that there exists some way to obtain the required Z_0. Often there are Z_0's obtained from Z by extended simple moves in M^3, that admit no further extended simple moves, and do not even have components with free fundamental group. This contrasts with Theorem 1 of [20] in which every Z_0^3 that admits no further extended simple moves satisfies the conclusion.

A piecewise-linear 3-manifold P^3 is *irreducible* if every polyhedral 2-sphere in Int P^3 bounds a 3-cell. P^3 is *prime* if every polyhedral, separating 2-sphere in Int P^3 bounds a 3-cell. If P_1^3 and P_2^3 are piecewise-linear 3-manifolds the *interior connected sum*, denoted by $P_1^3 \# P_2^3$, is obtained by choosing polyhedral 3-cells B_1^3, B_2^3 in Int P_1^3, Int P_2^3, respectively, removing Int B_1^3 and Int B_2^3, and identifying ∂B_1^3 and ∂B_2^3 by a piecewise-linear homeomorphism.

Lemma 9. *If R^3 is a compact, irreducible, piecewise-linear 3-manifold that has free fundamental group, then R^3 is a handlebody or a homotopy 3-sphere.*

Proof. Suppose R^3 is closed. The only incompressible, 2-sided surfaces that R^3 can contain are 2-spheres. Theorem 1 of [27] shows $\pi_1(R^3)$ is trivial. Hence R^3 is a homotopy 3-sphere.

Suppose $\partial R^3 \neq \emptyset$ and ∂R^3 contains a surface other than a 2-sphere. By the loop theorem [26], R^3 admits a simple reduction to R_1 (possibly not connected).

Note each component of R_1 is irreducible and has free fundamental group. Hence there is a compact polyhedron Z such that each component of Z is a 3-cell and Z is obtained from R^3 by a finite number of simple reductions. Since R^3 is obtained from Z by adding a finite number of solid 1-handles, R^3 is a handlebody.

Theorem 10. *Suppose R^3 is a compact, piecewise-linear 3-manifold with free fundamental group. Then R^3 is piecewise-linearly homeomorphic to an interior connected sum of a finite number of handlebodies, irreducible homotopy 3-spheres, and S^2 bundles over S^1. (S^n denotes the n-sphere.)*

Proof. The finiteness theorem [7], [8] implies that R^3 is piecewise-linearly homeomorphic to a finite interior connected sum $P_1^3 \# \cdots \# P_k^3$ of prime 3-manifolds. The proof of Lemma 1 of [24] shows that any prime 3-manifold that is not irreducible is an S^2 bundle over S^1. Since the fundamental group of R^3 is isomorphic to the free product of the fundamental groups of $P_1^3, \cdots, P_k^3$, each P_i^3 has free fundamental group. By Lemma 9 each irreducible P_i^3 is a handlebody or a homotopy 3-sphere.

4. **1-dimensional compact sets.** In this section we investigate the relationship between arc pushing properties, separation properties, and definability by handlebodies for compact sets of dimension no larger than one.

Suppose $\epsilon > 0$. A handlebody H^3 is *ϵ-thin* if there exists a collection $B_1, \cdots, B_l$ of polyhedral 3-cells such that $\bigcup_{i=1}^{l} B_i = H^3$, $B_i \cap B_j$ is either empty or a polyhedral 2-cell $D_{i,j}$ for $i \neq j$, $B_i \cap B_j \cap B_k = \emptyset$ for i, j, and k distinct, and each B_i has diameter less than ϵ.

Remarks. (1) The union of any subcollection of $B_1, \cdots, B_l$ is a collection of ϵ-thin handlebodies. (2) If T is a triangulation of a 3-manifold of mesh less than ϵ and L is a 1-dimensional subcomplex of T, then a second derived regular neighborhood of L yields a 2ϵ-handlebody. The required B_i correspond to the stars of the vertices of the induced first derived subdivision of L. (3) Suppose G is a polyhedral, finite graph in Int H^3 such that $G \cap D_{i,j}$ is exactly one point for all $D_{i,j}$'s and we can consider B_i to be a piecewise-linear cone over ∂B_i such that $G \cap B_i$ corresponds to a piecewise-linear subcone over $G \cap \partial B_i$. Then H^3 ϵ-retracts onto G by a retraction that maps B_i into B_i for all i. (4) Suppose H^3 is a subpolyhedron of the interior of a 3-manifold M^3 and G is a graph as described in Remark (3). Suppose N is a regular neighborhood of G in Int H^3 such that $N \cap D_{i,j}$ is a 2-cell for all $D_{i,j}$. Let U be a neighborhood of H^3. Then there is a piecewise-linear ϵ-homeomorphism (i.e., moves no point as much as ϵ) f of M^3 onto itself such that $f(N) = H^3$, $f(N \cap D_{i,j}) = D_{i,j}$, and f is the identity off U.

Suppose X is a proper, compact subset of the interior of a piecewise-linear 3-manifold M^3. (Keep this notation for the remainder of the paragraph.) X is *definable by ϵ-thin handlebodies* if X is definable by 3-manifolds each component of which is an ϵ-thin handlebody. X is *definable by thin handlebodies* if, for each $\epsilon > 0$, X is definable by ϵ-thin handlebodies. X has the (*strong*) *arc pushing property* if, given $\epsilon > 0$ and a polyhedral arc A contained in Int M^3 such that $X \cap \partial A = \emptyset$, there is a piecewise-linear (ϵ)-homeomorphism h of M^3 onto M^3 such that h is the identity outside the ϵ-neighborhood of X and $h(A) \cap X = \emptyset$. (McMillan [16] has shown in orientable 3-manifolds the arc pushing property is equivalent to being definable by handlebodies. No essential change in his proof is needed to drop orientability. The arc pushing properties have also been investigated in [4], [5], [23]).

The proof of Lemma 11 is immediate from Remark (*1*).

Lemma 11. *Suppose X is a compact, proper subset of the interior of a piecewise-linear* 3-*manifold. Suppose Y is a closed subset of X. If X is definable by thin handlebodies, then Y is definable by thin handlebodies.*

Lemma 12. *Suppose X is a compact metric space. X has dimension less than or equal to one if and only if there is an embedding of X in E^3 that is definable by thin handlebodies.*

Proof. If $\dim(X) \leq 1$, then X embeds in M, Menger's universal 1-dimensional curve [1*1*]. Lefschetz [11, p. 529] embeds M in E^3 so that M is definable by thin handlebodies. (See Remark (2).) Hence by Lemma 11, X embeds in E^3 so as to be definable by thin handlebodies.

Using Remark (3), we see that if X is definable by ϵ-thin handlebodies, X ϵ-maps onto a finite graph. Hence X is at most 1-dimensional.

Theorem 13. *Suppose X is a compact, proper subset of the interior of a piecewise-linear* 3-*manifold M^3. The following statements are equivalent.*

A. *X is definable by thin handlebodies.*

B. *If p and q are distinct points in X there is a tame* (*i.e.*, *definable by* 3-*cells in M^3*), *compact*, 0-*dimensional subset of X that separates p and q.*

C. *X has the strong arc pushing property.*

Proof. We can assume M^3 is compact. In order to show statement A implies statement B, let p and q be distinct points of X. It is sufficient to find a tame 2-sphere in Int M^3 that separates p from q and intersects X in a 0-dimensional set. Let S be a polyhedral 2-sphere in Int M^3 that separates p and q. There is a $\delta > 0$ such that if h is a δ-homeomorphism of M^3 onto M^3, then $h(S)$ separates p and q. In view of the method for constructing homeomorphisms given in [2, Theorem 7.1], it is sufficient to show that given $\epsilon > 0$ and a polyhedral 2-sphere S_1 in Int M^3 there is a piecewise-linear ϵ-homeomorphism f of M^3 onto itself such that $f(S_1) \cap X$ has components of diameter less than ϵ. Suppose $\epsilon > 0$ and a polyhedral 2-sphere S_1 in Int M^3 are given. There exists a disjoint collection $H_1^3, \cdots, H_r^3$ of ϵ-thin handlebodies in Int M^3 whose union contains X. Let $U_1, \cdots, U_r$ be disjoint neighborhoods of $H_1^3, \cdots, H_r^3$, respectively. Let H^3 be one of $H_1^3, \cdots, H_r^3$ and U be the corresponding $U_1, \cdots, U_r$. Recall the notation used in defining H^3 as an ϵ-thin handlebody. We can assume that no $D_{i,j}$ is entirely contained in S_1. Hence we can pick a graph G, as in Remark (3), such that, for all $D_{i,j}$, $G \cap S_1 \cap D_{i,j} = \emptyset$. Now select a regular neighborhood N of G in H^3, as in Remark (4), such that $N \cap S_1 \cap D_{i,j} = \emptyset$ for all $D_{i,j}$'s. Hence there exists a piecewise-linear ϵ-homeomorphism g of M^3 onto itself such that g is fixed off U, $g(N) = H^3$, and $g(N \cap D_{i,j}) = D_{i,j}$. Therefore each component of $g(S_1) \cap H^3$ is contained in some B_i and hence has diameter less than ϵ. Let $g_1, \cdots, g_r$ be the homeomorphisms obtained for $H_1^3, \cdots, H_r^3$, respectively. Let f be the composition of $g_1, \cdots, g_r$. Each component of $f(S_1) \cap (\bigcup_{i=1}^r H_i^3)$ has diameter less than ϵ. So each component of $f(S_1) \cap X$ has diameter less than ϵ. Hence f is our required homeomorphism.

We will now show statement B implies statement C. Assume statement B holds. We can strengthen statement B by the following observations. The alternative proof of Theorem 6.1 of [2] and Theorem II 2 of [10] show a finite union of tame, 0-dimensional, compact subsets of Int M^3 is a tame, 0-dimensional compact subset. By simple compactness arguments we can establish the following two statements in order. If $p \in X$ and Y is a closed subset of X that misses p then there is a tame 0-dimensional compact subset P of X that separates p from all points of $Y - P$. If $\epsilon > 0$ there exists a tame, 0-dimensional, compact subset P_ϵ of X such that each component of $X - P_\epsilon$ has diameter less than ϵ.

Suppose $\epsilon > 0$ and A is a polygonal arc in Int M^3 such that $X \cap \partial A = \emptyset$. Since X contains no open subset of M^3, there is a piecewise-linear $\epsilon/3$-homeomorphism h of M^3 onto itself such that h is fixed off the $\epsilon/3$-neighborhood of X and each component of $h(A) \cap X$ has diameter less than $\epsilon/3$. Let $A_1, \cdots, A_t$ be disjoint subarcs of $h(A)$ of diameter less than $\epsilon/3$ such that $A_i \cap X \neq \emptyset$ and $X \subset \bigcup_{i=1}^t \operatorname{Int} A_i$. Let $U_1, \cdots, U_t$ be disjoint neighborhoods of $A_1, \cdots, A_t$, respectively, of diameter less than $\epsilon/3$. In the remainder of this proof our homeomorphisms will be fixed off $\bigcup_{i=1}^t U_i$. Note $\bigcup_{i=1}^t U_i$ is contained in the ϵ-neighborhood of X. Let $D_1, \cdots, D_t$ be polyhedral 2-cells such that $D_i \subset U_i$, $D_i \cap h(A) = A_i$ and $A_i \cap \partial D_i = \partial A_i$. Since ∂A_i misses X we can find disjoint subarcs A_i', A_i'' of ∂D_i such that A_i', A_i'' miss X and A_i', A_i'' each contain one endpoint of A_i. Let E_i', E_i'' be the closures of the two components of $\partial D_i - (A_i' \cup A_i'')$. Let δ be the minimum of the distance from E_i' to E_i'', $i = 1, \cdots, t$; the distance from X to $A_i' \cup A_i''$, $i = 1, \cdots, t$; the distance from X to the closure of $h(A) - \bigcup_{i=1}^t A_i$. There exists a piecewise-linear $\delta/3$-homeomorphism f of M^3 onto M^3 such that f is fixed off

$$\bigcup_{i=1}^{t} U_i, f\left(h(A) \cup \left(\bigcup_{i=1}^{t} D_i\right)\right) \cap X \subset \bigcup_{i=1}^{t} f(D_i - (A_i' \cup A_i'')),$$

and $f(\bigcup_{i=1}^{t} D_i)$ misses $P_{\delta/3}$. ($P_{\delta/3}$ is defined in the preceding paragraph.) Therefore each component of $f(D_i) \cap X$ has diameter less than $\delta/3$. Note no component of $f(D_i) \cap X$ separates $f(A_i')$ from $f(A_i'')$ in $f(D_i)$ since $f(E_i')$ and $f(E_i'')$ are at least a distance $\delta/3$ apart. In $f(D_i)$ select a polyhedral arc A_i^* such that $A_i^* \cap f(\partial D_i) = f(\partial A_i)$ and A_i^* misses X. There is a piecewise-linear $\epsilon/3$-homeomorphism g such that g is fixed off $\bigcup_{i=1}^{t} U_i$, $gf(A_i) = A_i^*$ and g is fixed on $f(h(A) - (\bigcup_{i=1}^{t} \text{Int } A_i))$. Now gfh is a piecewise-linear ϵ-homeomorphism fixed off the ϵ-neighborhood of X such that $gfh(A) \cap X = \emptyset$.

In conclusion we will show statement C implies statement A. Let U be a neighborhood of X in M^3 and suppose $\epsilon > 0$. Let W be a compact polyhedron in U such that each component of W is a 3-manifold and $X \subset \text{Int } W$. Suppose T is a triangulation of W such that the mesh of T is less than $\epsilon/18$ and T'' is a second derived subdivision of T such that no new vertex of T' lies in X. (X contains no open set.) Let $T^{(1)}$ be the 1-skeleton of T and $T_{(1)}$ be the dual 1-skeleton (i.e., the collection of all 1-simplexes and vertices of T' that miss $T^{(1)}$). Let H^1, H_1 be the simplicial neighborhoods of $T^{(1)}, T_{(1)}$, respectively, in T''. Then $H^1 \cup H_1 = W$, $H^1 \cap H_1 \subset \partial H^1 \cap \partial H_1$ and H^1, H_1 are collections of $\epsilon/9$-thin handlebodies. (See Remark (2).) Let $A_1, \cdots, A_s$ be the 1-simplexes of $T_{(1)}$. The endpoints of $A_1, \cdots, A_s$ miss X. Apply the strong arc pushing property to inductively choose ϵ_i, h_i, $1 \leq i \leq s$; such that ϵ_i is less than $\epsilon/9s$, ϵ_i is less than the distance from X to $h_{i-1} \cdots h_1(A_1 \cup \cdots \cup A_{i-1})$, ϵ_i is less than the distance from X to $\partial A_{i+1} \cup \cdots \cup \partial A_s$, ϵ_i is less than the distance from X to $h_{i-1} \cdots h_1(\partial W) = \partial W$, and h_i is a piecewise-linear ϵ_i-homeomorphism of M^3 onto itself such that $h_i h_{i-1} \cdots h_1(A_i) \cap X = \emptyset$. Let $h = h_s \cdots h_1$. Then h is a piecewise-linear $\epsilon/9$-homeomorphism of M^3 onto M^3 such that $h(W) = W$ and $h(T_{(1)}) \cap X = \emptyset$. Note $h(H^1), h(H_1)$ are collections of $\epsilon/3$-thin handlebodies. Let N be a nice regular neighborhood of $h(T_{(1)})$ in $h(H_1)$ that misses X. There is a piecewise-linear $\epsilon/3$-homeomorphism f of M^3 onto M^3 such that $fh(H_1)$ misses X, $fh(\text{Int } W)$ contains X and $fh(W) \subset U$. (The inverse of the type of homeomorphism promised in Remark (4) is almost our f. The situation is slightly changed since $T_{(1)}$ intersects ∂H_1.) Hence $fh(H^1)$ is a collection of ϵ-thin handlebodies contained in U that contain X. The proof of Theorem 13 is complete.

Corollary 14. *There is an embedding h of Menger's universal 1-dimensional curve M in E^3 and distinct points p and q in $h(M)$ such that* (1) *every 0-dimensional compact subset of $h(M)$ that separates p and q is wild, and* (2) *every tame surface in E^3 that separates p and q must intersect $h(M)$ in a 1-dimensional set.*

Proof. The embeddings of M given in [4], [5], [23] all suffice in view of Theorem 13. Bothe [5] even gives an embedding definable by handlebodies.

5. Main theorem and corollaries. Our main result is the following theorem.

Theorem 15. *Suppose m and l are nonnegative integers and that p is either zero or a prime. Let Y be a compact metric space such that each component of Y has properties* $\mathrm{uv}(1, m; Z_p)$, $\mathrm{uv}(2, l; Z_p)$ *and* UVF. *Suppose X is a compact metric space and* $\mathrm{Sh}(Y) \geq \mathrm{Sh}(X)$.

Let h be a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. If M^3 is not orientable assume $p = 2$. Then $h(X) = \bigcap_{i=1}^{\infty} H_i$ where H_i is a compact polyhedron in $\mathrm{Int}\, M^3$ *such that each component of H_i is a 3-manifold with free fundamental group and at most $l+1$ boundary components, and* $H_{i+1} \subset \mathrm{Int}\, H_i$.

Proof. We can assume M^3 is compact. Suppose $h(X) \subset U \subset M^3$ where U is an open set. It is sufficient to find a compact polyhedron $H \subset U$ such that each component of H is a 3-manifold with free fundamental group and at most $l+1$ boundary components, and such that $h(X) \subset \mathrm{Int}\, H$. Note that Theorem 1 and Lemma 2 imply that $h(X)$ has properties $\mathrm{uv}(1, m; Z_p)$, $\mathrm{uv}(2, l; Z_p)$, and UVF.

Recalling the remark in §2 and Theorem 3 we can find compact polyhedra $W_0, \cdots, W_k$, whose components are 3-manifolds, such that

(1) $W_{i+1} \subset \mathrm{Int}\, W_i$, $i = 0, \cdots, k-1$, $X \subset \mathrm{Int}\, W_k$, $W_0 \subset U$,

(2) each component of W_k has at most $l+1$ boundary components,

(3) for each component W'_{i+1} of W_{i+1} the inclusion induced homomorphism $\pi_1(W'_{i+1}) \to \pi_1(W'_i)$ factors through a free group, $i = 0, \cdots, k-1$ (W'_i is the component of W_i that contains W'_{i+1}), and

(4) $k > (\rho H_2(M^3; Z_2) + l + 1)2m + \rho H_1(M^3; Z_2) + 1$

By Theorem 7 we can find compact polyhedra Z and Z_0 such that each component of Z is a 3-manifold, $h(X) \subset Z \subset \mathrm{Int}\, W_k$, Z_0 is obtained from Z by simple moves in W_k, and every component T of ∂Z_0 satisfies $\rho H_1(T; Z_p) = \rho H_1(T; Z_2) \leq 2m$. Let $L_1, \cdots, L_s$ be the components of $W_k - \mathrm{Int}\, Z_0$ that do not intersect ∂W_k. Let $V = Z_0 \cup L_1 \cup \cdots \cup L_s \subset \mathrm{Int}\, W_k$. Note $\partial V \subset \partial Z_0$.

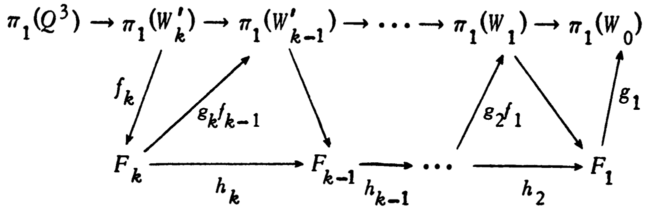
Our goal in the next few paragraphs is to show each component of V has free image in some W_i. Let Q^3 be a component of V and W'_k be the component of W_k that contains Q^3. Since W'_k has at most $l+1$ boundary components, $W'_k - \mathrm{Int}\, Q^3$ has at most $l+1$ components. Hence $M^3 - \mathrm{Int}\, Q^3$ has at most $l+1$ components.

We will now show Q^3 has at most $\rho H_2(M^3; Z_2) + l + 1$ boundary components. Suppose $S_1, \cdots, S_t$ are the boundary components of Q^3, $M^3 - \mathrm{Int}\, Q^3$ has $r \leq l+1$ components, and $S_1, \cdots, S_t$ are numbered so that $S_{t-r+1}, \cdots, S_t$ lie in different components of $M^3 - \mathrm{Int}\, Q^3$. Since $M^3 - (S_1 \cup \cdots \cup S_{t-r})$ is connected, $S_1, \cdots,$

S_{t-r} are carriers for linearly independent elements of $H_2(M^3; Z_2)$. So $t-r \le \rho H_2(M^3; Z_2)$ and $t \le \rho H_2(M^3; Z_2) + l + 1$ as asserted. Hence $\rho H_1(\partial Q^3; Z_2) \le (\rho H_2(M^3; Z_2) + l + 1)2m$. Using the Mayer-Vietoris sequence for Q^3 and the closure of $M^3 - Q^3$ with coefficients Z_2, we see that image$[H_1(\partial Q^3; Z_2) \to H_1(Q^3, Z_2)]$ contains $\ker[H_1(Q^3; Z_2) \to H_1(M^3; Z_2)]$. Hence

$$\begin{aligned} \rho H_1(Q^3; Z_2) &\le \rho \text{ image } [H_1(Q^3; Z_2) \to H_1(M^3; Z_2)] \\ &\quad + \rho \ker [H_1(Q^3; Z_2) \to H_1(M^3; Z_2)] \\ &\le \rho H_1(M^3; Z_2) + \rho H_1(\partial Q^3; Z_2) < K - 1. \end{aligned}$$

We will use primes to denote components. There are components $W'_0, \cdots, W'_k$ of $W_0, \cdots, W_k$ such that $Q^3 \subset W'_k \subset \cdots \subset W'_0$. Recall that $\pi_1(W'_i) \to \pi_1(W'_{i-1})$ factors through a free group F_i. Let $f_i: \pi_1(W'_i) \to F_i$ and $g_i: F_i \to \pi_1(W'_{i-1})$ be factoring homomorphisms. The following diagram is consistent. We define $h_i = f_{i-1}g_i: F_i \to F_{i-1}$.



Let $I_i = \text{image}[\pi_1(Q^3) \to F_i]$, $i = 1, \cdots, k$. Let $\theta_i = h_i|_{I_i}$. Note that θ_i, $i = 2, \cdots, k$, is an epimorphism. Hence $\rho I_k \ge \rho I_{k-1} \ge \cdots \ge \rho I_1 \ge 0$. Consider the consistent diagram

$$\begin{array}{ccc} \pi_1(Q^3) \longrightarrow \pi_1(W'_k) & \xrightarrow{f_k} & I_k \\ \downarrow & & \downarrow \\ H_1(Q^3) & \longrightarrow & J_k \\ \downarrow & & \downarrow \\ H_1(Q^3; Z_2) & \longrightarrow & J_k \otimes Z_2 \end{array}$$

where the top vertical arrows represent abelianization epimorphisms, J_k is the abelianization of I_k, the middle horizontal arrow is the epimorphism induced by the top half of the diagram, the bottom vertical arrows are tensor product epimorphisms, and the bottom horizontal arrow is the epimorphism induced from the diagram. Since I_k is a free group, $\rho I_k = \rho J_k = \rho J_k \otimes Z_2$. Therefore, $\rho I_k \le \rho H_1(Q^3; Z_2) < k - 1$. It follows that, for some i, I_i and I_{i-1} are free groups on the same number of generators. So by [12, p. 312] θ_i is an isomorphism, $g_i|_{I_i}$ is a monomorphism, and so Q^3 has free image in W'_{i-1}.

Now we can apply Theorem 8 to find a V_0 obtained from V by extended simple moves in U such that each component of V_0 has free fundamental group. Recall that $h(X) \subset Z \subset U$, Z_0 is obtained from Z by simple moves in U, and $Z_0 \subset V$. By applying [20, Theorem 1$^+$] we can find a compact polyhedron R in U such that each component of R is a 3-manifold with free fundamental group and $h(X) \subset \operatorname{Int} R$. Hence $h(X)$ is definable by free 3-manifolds.

The R we obtained in the preceding paragraph satisfies the conditions we require for H except possibly some component of R may have more than $l+1$ boundary components. Keep the notation of the preceding paragraphs. Let P be a compact polyhedron in $\operatorname{Int} R$ such that each component P' of P is a 3-manifold with free fundamental group, $h(X) \subset \operatorname{Int} P$, and ρ image$[H_2(P'; Z_p) \to H_2(R'; Z_p)] \leq l$ where R' is the component of R that contains P'.

Using Theorem 10 we see that P' is homeomorphic to $P_1 \# \cdots \# P_r$ where each P_i is either a homotopy 3-sphere, S^2 bundle over S^1, or a handlebody. Let $T_1, \cdots, T_{r-1}$ be disjoint, polyhedral 2-spheres in P' such that the components of P' cut along $T_1, \cdots, T_{r-1}$ are equivalent, in the sense of Milnor [24], to $P_1, \cdots, P_r$. Note that if J is a boundary component of P', then J plus some subcollection of $T_1, \cdots, T_{r-1}$ bounds a compact 3-manifold $N_J \subset P'$. (N_J is homeomorphic to a handlebody minus the interiors of a disjoint collection of 3-cells.)

Let C be a small regular neighborhood of $T_1 \cup \cdots \cup T_{r-1}$ in P'. Let $U_1, \cdots, U_p$ be the components of $R' - \operatorname{Int} C$ and $V_1, \cdots, V_q$ be the components of $P' - \operatorname{Int} C$. Note $\partial U_i \cap \operatorname{Int} R'$ and $\partial V_j \cap \operatorname{Int} P'$ consist of 2-spheres. Hence each U_i and V_j has free fundamental group. Let D be C union the U_i's that miss $\partial R'$. Each component of D has free fundamental group and 2-sphere boundary components. Let H' be D union the V_j's that miss $\operatorname{Int} D$. Note H' is connected, $\pi_1(H')$ is free, $\partial H' \subset \partial P'$, and $H' \supset P'$. If N is a component of $R' - \operatorname{Int} H'$ that misses $\partial R'$, let $J_1, \cdots, J_t$ be the components of $N \cap H'$. Then $N \cup N_{J_1} \cup \cdots \cup N_{J_t}$ contains some U_i that should have been added to C to obtain D. Hence each component of $R' - \operatorname{Int} H'$ must intersect $\partial R'$. Note also that if P'' is another component of P then H' contains P'' or $H' \cap P'' = \emptyset$. Now the last three paragraphs of the proof of Theorem 3 apply to give us our required H. The proof of Theorem 15 is complete.

A (*homotopy cube-with-handles*) *homotopy handlebody* is a 3-manifold homeomorphic to the interior connected sum of a homotopy 3-sphere and a (cube-with-handles) handlebody. The meaning of being *definable by homotopy handlebodies* (*or homotopy cubes-with-handles*) should be clear.

If X is a compact metric space Borsuk [3] says the *fundamental dimension* of X is less than or equal to one, written $\mathrm{Fd}(X) \leq 1$, if there exists an at most 1-dimensional compact metric space Y such that $\mathrm{Sh}(Y) \geq \mathrm{Sh}(X)$.

Corollary 16. *Suppose m is a nonnegative integer and $p = 0$ or is a prime. Suppose X is a compact metric space such that each component of X has properties* $\mathrm{uv}(1, m; Z_p)$, $\mathrm{uv}(2, 0; Z_p)$ *and* UVF. *Suppose h is a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. If M^3 is nonorientable assume $p = 2$. Then* (1) *$h(X)$ is definable by homotopy handlebodies and* (2) *If M^3 is orientable, $h(X)$ is definable by homotopy cubes-with-handles.*

Proof. (2) follows from (1). By Theorem 15 we know that $h(X)$ is definable by free 3-manifolds each component of which has connected boundary. Let U be a neighborhood of $h(X)$ in M^3 such that ρ image$[H_2(U; Z_p) \to H_2(M^3; Z_p)] = 0$. It is sufficient to show that any compact piecewise-linear 3-manifold contained in U with free fundamental group and connected boundary must be a homotopy handlebody. Suppose W^3 is a compact, piecewise-linear 3-manifold in U with $\pi_1(W^3)$ free and ∂W^3 connected. By Theorem 10, if W^3 is not a homotopy handlebody then W^3 contains a nonseparating, polyhedral 2-sphere S. But $S \subset U$ and S is the carrier of a representative of a nontrivial element of $H_2(M^3; Z_p)$. Our choice of U makes this impossible. Hence W^3 must be a homotopy handlebody.

Corollary 17. *Let m be a nonnegative integer and let $p = 0$ or a prime. Suppose X is a compact metric space such that each component of X has property* $\mathrm{uv}(1, m; Z_p)$. *Let h be a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. If M^3 is nonorientable let $p = 2$. Then* (1) *if the dimension of X is ≤ 1, $h(X)$ is definable by handlebodies, and* (2) *if* $\mathrm{Fd}(X) \leq 1$, *then $h(X)$ is definable by homotopy handlebodies.*

Proof. If the dimension of X is ≤ 1, by Lemma 12 we can embed X in E^3 so as to be definable by thin handlebodies. Hence X has properties $\mathrm{uv}(2, 0; Z_p)$ and UVF. By Corollary 16, $h(X)$ is definable by homotopy handlebodies. Using Theorem 4 of [23] we see that we can choose the homotopy handlebodies to be handlebodies.

If $\mathrm{Fd}(X) \leq 1$ there exists a compact metric space Y such that $\dim(Y) \leq 1$ and $\mathrm{Sh}(Y) \geq \mathrm{Sh}(X)$. By Lemma 12, Y has properties UVF and $\mathrm{uv}(2, 0; Z_p)$. Hence by Theorem 1, X has the same properties. Using Corollary 16 we see that $h(X)$ is definable by homotopy handlebodies. The proof is complete.

A *continuum* is a compact connected metric space. A topological space is *locally n-connected*, $n \geq 0$, if the space is locally connected in dimension i for $0 \leq i \leq n$. See [9].

Lemma 18. *Let $p = 0$ or a prime and let n be a nonnegative integer. Suppose*

X is a locally n-connected continuum contained in the interior of a nonclosed, piecewise-linear 3-manifold M^3. Then there exists a compact, polyhedral 3-manifold N^3 such that

$$X \subset \operatorname{Int} N^3, \qquad N^3 \subset \operatorname{Int} M^3,$$

$$\text{image}\,[H_i(X; Z_p) \to H_i(M^3; Z_p)] = \text{image}\,[H_i(N^3; Z_p) \to H_i(M^3; Z_p)]$$

$$\textit{for } 0 \leq i \leq n+1,$$

and

$$\text{image}\,[\pi_1(X) \to \pi_1(M^3)] = \text{image}\,[\pi_1(N^3) \to \pi_1(M^3)].$$

Proof. We may assume M^3 is compact. Since M^3 is an ANR there is an $\epsilon > 0$ such that any two ϵ-close maps from a compact metric space into M^3 are homotopic. There exists a $\delta > 0$ such that if N^3 is a compact polyhedral 3-manifold in the δ-neighborhood of X and $\sigma = \Sigma\, n_j\, \sigma_j$ is a δ-small i-cycle in N^3 (i.e., each singular i-simplex σ_j has diameter less than δ, $\sigma_j \subset N^3$, n_j belongs to Z_p, and σ represents an element of $H_i(N^3; Z_p)$) there is a singular i-cycle $\sigma' = \Sigma n_j \sigma_j'$ in X such that σ_j and σ_j' are ϵ-close for each j. (See Theorem 4.1 of Chapter 5 of [9].) Hence σ and σ' represent the same element of $H_i(M^3; Z_p)$. We leave the proof of the condition on images of fundamental groups to the reader.

Addendum. *If, for some $i \leq n+1$, $H_i(X; Z_p)$ is finitely generated, then X has property* uv$(i, m; Z_p)$ *where $m = \rho H_i(X; Z_p)$.*

The last two corollaries of Theorem 15 are stated only in terms of a continuum. Of course there are similar corollaries where we apply the hypotheses to the components of a compact set.

Corollary 19. *Suppose X is a compact* ANR *and h is a topological embedding of X into the interior of a nonclosed, piecewise-linear 3-manifold M^3. Let $p = 0$ or a prime. If M^3 is nonorientable assume $p = 2$. Then* (1) *$h(X)$ is definable by free 3-manifolds if and only if $\pi_1(X)$ is free, and* (2) *$h(X)$ is definable by homotopy handlebodies if and only if $\pi_1(X)$ is free and X has property* uv$(2, 0; Z_p)$.

Proof. Let N^3 be a compact, polyhedral 3-manifold in Int M^3 such that $h(X) \subset \operatorname{Int} N^3$ and N^3 retracts onto $h(X)$. Note the inclusion $h(X) \subset N^3$ induces monomorphisms on homology and homotopy groups. We will prove (1). The proof of (2) is similar. If $h(X)$ is definable by free 3-manifolds then $\pi_1(X)$ is free. By Lemma 18 and the Addendum, X has properties uv$(1, m; Z_p)$ and uv$(2, l; Z_p)$ for $m = \rho H_1(X; Z_p)$ and $l = \rho H_2(X; Z_p)$. Again by Lemma 18 we see X has property UVF. By Theorem 15 we are done.

Corollary 20. *Suppose X is a continuum and h_1, h_2 are topological embeddings of X into the interiors of nonclosed, piecewise-linear 3-manifolds M_1^3, M_2^3, respectively. Let $p = 0$ or a prime. If either M_1^3 or M_2^3 is nonorientable, assume $p = 2$. Then* (1) *If X is locally 1-connected and $H_1(X; Z_p)$ and $H_2(X; Z_p)$ are finitely generated, then $h_1(X)$ is definable by free 3-manifolds if and only if $h_2(X)$ is definable by free 3-manifolds, and* (2) *If X is locally 0-connected and $H_1(X; Z_p)$ is finitely generated, then $h_1(X)$ is definable by homotopy handlebodies if and only if $h_2(X)$ is definable by homotopy handlebodies.*

Proof. Considering (1) we see that by the addendum to Lemma 18, X has properties uv(1, m; Z_p) and uv(2, l; Z_p) for $m = \rho H_1(X; Z_p)$ and $l = \rho H_2(X; Z_p)$. If either $h_1(X)$ or $h_2(X)$ is definable by free 3-manifolds, then X has property UVF. By Theorem 15, (1) holds.

For (2) note that X has property uv(1, m; Z_p) where $m = \rho H_1(X; Z_p)$. If either $h_1(X)$ or $h_2(X)$ is definable by homotopy handlebodies then X has properties UVF and uv(2, 0; Z_p). Hence by Corollary 16, (2) holds.

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